MATH 245 F22, Exam 1 Solutions

1. Carefully define the following terms: prime, converse.

   Let n be an integer with $n \geq 2$. We call n prime if there does not exist an integer a with both $1 < a < n$ and $a|n$. For any propositions p, q, the converse of conditional proposition $p \to q$ is the proposition $q \to p$.

2. Carefully state the following theorems: Distributivity theorem (for propositions), Conditional Interpretation theorem.

   The Distributivity theorem says: for any propositions p, q, r, we have $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. The Conditional Interpretation theorem states: for any propositions p, q, we have $p \to q \equiv q \vee \neg p$.

3. Let $a, b \in \mathbb{N}_0$. Suppose that $a \leq b$. Prove that $a^2 \leq b^2$.

   Since $a \leq b$, we have $b - a \in \mathbb{N}_0$. Since $b, a \in \mathbb{N}_0$, their sum $b + a$ is also in $\mathbb{N}_0$. Since $b - a$ and $b + a$ are each in $\mathbb{N}_0$, their product $(b - a)(b + a) = b^2 - a^2$ is also in $\mathbb{N}_0$. Hence $a^2 \leq b^2$.

4. Prove or disprove: For all integers a, b, c, if $ac|bc$ then $a|b$.

   The statement is false. To disprove we need a counterexample, i.e. specific choices for a, b, c where $ac|bc$ and yet $a \nmid b$. Many answers are possible, e.g. $a = 2, b = 3, c = 0$, but all will have $c = 0$. Now $ac|bc$ because taking $k = 5$ we have $ack = (2)(0)(5) = 0 = (3)(0) = bc$. However, $a \nmid b$ since if $at = b$ we have $2t = 3$, so $t = \frac{3}{2} \notin \mathbb{Z}$.

5. Prove or disprove: For all integers n, if $7 \nmid n$ then $14 \nmid n$.

   The statement is true, and we will use a contrapositive proof. Let $n \in \mathbb{Z}$ be arbitrary. Suppose that $14|n$. Then there is some integer k with $14k = n$. We now write $7(2k) = n$. Since $2k \in \mathbb{Z}$, we have $7|n$.

6. Carefully state the double negation theorem and prove it without truth tables.

   The double negation theorem says: For all propositions p, we have $\neg\neg p \equiv p$. Proof: Let p be an arbitrary proposition. If p is T, then $\neg p$ is F and $\neg\neg p$ is T. On the other hand, if p is F, then $\neg p$ is T and $\neg\neg p$ is F. In both cases, p and $\neg\neg p$ agree.

7. Find a well-formed expression with three bound and two free variables.

   Many solutions are possible; all contain five variables altogether. Here is one possible answer, with integers a, b, c, x, y: $\forall a, \ \forall b, \ \forall c, \ a + b + c = x + y$.

8. Simplify the proposition $\neg \forall x \in \mathbb{R}, \ \exists y \in \mathbb{R}, \ \exists z \in \mathbb{Z} \ \ (x < y) \to (x < z \leq y)$ as much as possible, where only basic propositions are negated. Be sure to justify each step. DO NOT TRY TO PROVE OR DISPROVE THE RESULT, ONLY SIMPLIFY.

   Step 1: pull negation into quantifiers: $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ \neg\big((x < y) \to (x < z \leq y)\big)$
   Step 2: apply Negated CI thm (2.16): $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ (x < y) \wedge \neg(x < z \leq y)$
   Step 3: interpret double inequality: $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ (x < y) \wedge \neg\big((x < z) \wedge (z \leq y)\big)$
   Step 4: apply De Morgan's law: $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ (x < y) \wedge \big(\neg(x < z)\big) \vee \big(\neg(z \leq y)\big)$
   Step 5: negate inequalities: $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ (x < y) \wedge \big((x \geq z) \vee (z > y)\big)$
   Step 6 (optional): distributivity: $\exists x \in \mathbb{R}, \ \forall y \in \mathbb{R}, \ \forall z \in \mathbb{Z} \ \ (y > x \geq z) \vee (z > y > x)$

9. Without using truth tables, prove the "Composition Theorem":
For all propositions p, q, r, we have $(p \to q) \land (p \to r) \vdash p \to (q \land r)$.

All correct proofs begin by letting p, q, r be arbitrary propositions, and continue by assuming $(p \to q) \land (p \to r)$.

METHOD 1: Apply conditional interpretation twice to get $(q \lor \neg p) \land (r \lor \neg p)$. Then apply distributivity to get $(q \land r) \lor \neg p$. Then apply conditional interpretation again to get $p \to (q \land r)$.

METHOD 2: Two cases, depending on p.
Case 1: p is F. Then $\neg p$ is T, so by addition $(q \land r) \lor \neg p$.
Case 2: p is T. By simplification on the hypothesis twice, $p \to q$ and $p \to r$. By modus ponens twice, we get q and r. By conjunction, $q \land r$. By addition, $(q \land r) \lor \neg p$.
In both cases, $(q \land r) \lor \neg p$. Now we apply conditional interpretation to get $p \to (q \land r)$.

METHOD 3: Two cases, depending on p.
Case 1: p is F. Then $p \to (q \land r)$ is vacuously true.
Case 2: p is T. By simplification on the hypothesis twice, $p \to q$ and $p \to r$. By modus ponens twice, we get q and r. By conjunction, $q \land r$. Then $p \to (q \land r)$ is trivially true.
In both cases, $p \to (q \land r)$ is true.

Note: $p \to (q \land r)$ should be the dramatic conclusion, and should not appear earlier.

10. Using $p, q, \uparrow$ (possibly multiple times), but no other operators, find a proposition which is logically equivalent to $p \to q$. Justify your answer.

Various answers are possible, such as $p \uparrow (q \uparrow q)$ or $p \uparrow (p \uparrow q)$; this solution uses the former. The last two columns of the truth table below agree, which proves that $p \uparrow (q \uparrow q) \equiv p \to q$.

p	q	$q \uparrow q$	$p \uparrow (q \uparrow q)$	$p \to q$
T	T	F	T	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T